Descriptive complexity of diameter 2 properties

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Methods in Banach Spaces
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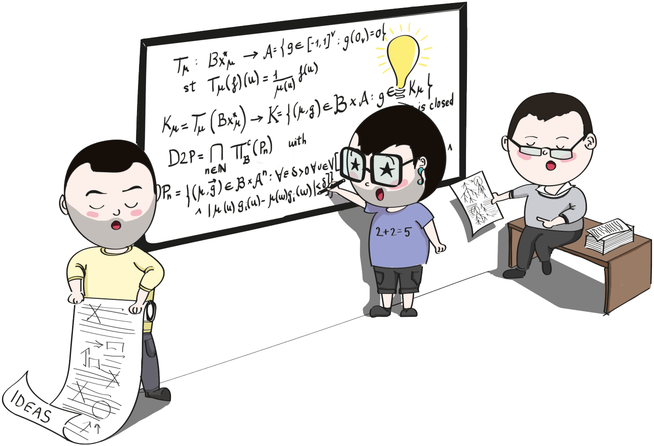
with
is closed
2+2=5
IDEAS

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Problem:

No canonical topology $\Rightarrow$ No distinction between Borel classes

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$(X_\mu, \hat{\mu})$ is a separable Banach space, V countable and dense.

Extensions of Bossard's idea

Given a separable Banach space $(X, \|\cdot\|)$ with a dense sequence $\{x_n\}$, we can obtain a seminorm $\mu \in \mathcal{P}$ such that $X \equiv X_\mu$ by defining for every $v = \sum a_n e_n \in V$ (with $\{e_n\}$ the canonical basis of c_{00})

$$\mu(v) = \left\| \sum_{n=1}^{\infty} a_n x_n \right\|.$$

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$$\mu(v) = \left\| \sum_{n=1}^{\infty} a_n x_n \right\| .$$

$$\mathcal{P}_\infty = \{\mu \in \mathcal{P} : X_\mu \text{ is infinite-dimensional}\}$$
$$\mathcal{B} = \{\mu \in \mathcal{P}_\infty : \bar{\mu} \text{ is a norm}\}$$

Extensions of Bossard's idea

Finite representability

If $\mathcal{F}$ is a class of Banach spaces, we say that a Banach space X is finitely representable in $\mathcal{F}$ if for any $\varepsilon > 0$ and any finite-dimensional subspace E of X, there exists a $Y \in \mathcal{F}$ and a finite-dimensional subspace G of Y such that it is $(1+\varepsilon)$-isomorphic to E.

Extensions of Bossard's idea

Theorem [CDDK22b, Proposition 2.9]

Let $\mathcal{F} \subset \mathcal{I}$ closed by isometries, that is, if $\mu \in \mathcal{F}$ and $\nu \in \mathcal{I}$ with $X_\nu \equiv X_\mu$, then $\nu \in \mathcal{F}$. Then

$$\mathrm{cl}_{\mathcal{I}}(\mathcal{F}) = \{\mu \in \mathcal{I} : X_\mu \text{ is finitely representable in } \mathcal{F}\}$$

where $\mathcal{I} \in \{\mathcal{P}, \mathcal{P}_\infty, \mathcal{B}\}$.

Diameter 2 properties

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A Banach space X has:

- LD2P: Every slice of B_X has diameter 2.
- D2P: Every w-open set of B_X has diameter 2.

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1. Give a purely geometrical characterization of the properties.
2. Find a way to “talk about the dual space”.

Codification of the dual space in $\mathcal{B}$

Given $\mu \in \mathcal{B}$ we define the map

$$
\begin{aligned}
T_\mu : B_{X_\mu^*} &\longrightarrow \ [-1,1]^V \\
f &\longmapsto T_\mu(f) : V \longrightarrow [-1,1] \\
& \qquad\qquad u \longmapsto \begin{cases} 0 & \text{if} \quad u = 0_V \\ \dfrac{1}{\mu(u)} f(u) & \text{if} \quad u \neq 0_V \end{cases}
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We will denote the image of this map by K_μ. Observe then that given $g \in [-1,1]^V$ we have that

$$g \in K_\mu \leftrightarrow g(0_V) = 0 \wedge \exists f \in B_{X_\mu^*} \forall u \in V \, f(u) = \mu(u) g(u).$$

Proof (sketch) that D2P is G_δ-complete in $\mathcal{B}$

Let's denote the isometry class of the spaces with the D2P as

$$\widehat{\text{D2P}} = \{\mu \in \mathcal{B} : X_\mu \text{ has the D2P}\}$$

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$$P_n = \Big\{(\mu, \vec{g}) \in \mathcal{B} \times \left([-1,1]^V\right)^n : \forall \varepsilon > 0 \, \forall \delta > 0 \, \forall u \in V \big[\mu(u) > 1 \vee \exists i \in \{1, \cdots, n\} \, g_i \notin K_\mu \vee \exists v, w \in V \big(\mu(v-w) > 2 - \varepsilon \wedge \mu(v), \mu(w) < 1 \wedge \forall i \in \{1, \cdots, n\} \big[|\mu(u)g_i(u) - \mu(v)g_i(v)| < \delta \wedge |\mu(u)g_i(u) - \mu(w)g_i(w)| < \delta\big]\big)\big]\Big\}.$$

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We have that

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The last formula is equivalent to

$$\widehat{\mathrm{D2P}} = \bigcap_{n=1}^{\infty} \pi^c_{\mathcal{B}}(P_n)$$

where

$$\pi^c_{\mathcal{B}}(P_n) = \left(\pi_{\mathcal{B}}\left(P_n^c\right)\right)^c$$

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In general, the set

$$K = \Big\{(\mu,g) \in \mathcal{B} \times [-1,1]^V : g \in K_\mu\Big\}$$

is closed.

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If $\widehat{\mathrm{D2P}}$ is F_σ in $\mathcal{B}$, then $\mathcal{B} \setminus \widehat{\mathrm{D2P}}$ is G_δ and dense. By Baire's theorem $\mathcal{B} \setminus \widehat{\mathrm{D2P}}$ has to intersect the isometry class of the Gurariĭ space, but this is a contradiction because the Gurariĭ space has de D2P.

Compendium of results

	LD2P	D2P	SD2P
$\mathcal{B}$	G_δ-complete	G_δ-complete	G_δ-complete
$\mathcal{P}_\infty$	G_δ-complete	$F_{\sigma\delta}$	G_δ-complete

	DLD2P	DD2P	DP
$\mathcal{B}$	G_δ-complete	G_δ-complete	G_δ-complete
$\mathcal{P}_\infty$	G_δ-complete	$F_{\sigma\delta}$	G_δ-complete

	LOH	WOH	OH
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Thank you!