Descriptive complexity of some geometric classes

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Borel hierarchy

Polish space: A separable completely metrizable topological space.

We denote by $\Sigma_1^0(X)$ the open sets of X and by $\Pi_1^0(X)$ its closed sets. Then:

$$\Sigma_\alpha^0(X) = \left\{ \bigcup_{n\in\mathbb{N}} A_n : \forall n \in \mathbb{N}, A_n \in \bigcup_{0<\beta<\alpha} \Pi_\beta^0(X) \right\},$$

$$\Pi_\alpha^0(X) = \left\{ \bigcap_{n\in\mathbb{N}} A_n : \forall n \in \mathbb{N}, A_n \in \bigcup_{0<\beta<\alpha} \Sigma_\beta^0(X) \right\}.$$

We have the **Borel hierarchy** for X, where $\Delta_\alpha^0(X) = \Sigma_\alpha^0(X) \cap \Pi_\alpha^0(X)$:

$$\begin{array}{ccccccccc} & \Sigma_1^0(X) & & \Sigma_2^0(X) & & \Sigma_3^0(X) & & & \cdots \\ \Delta_1^0(X) & & \Delta_2^0(X) & & \Delta_3^0(X) & & \Delta_4^0(X) & & \\ & \Pi_1^0(X) & & \Pi_2^0(X) & & \Pi_3^0(X) & & & \cdots \end{array}$$

Analytic set: $A \subset X$ is analytic if it is the continuous image of a polish space over X.

Seminorms model

V is the space of sequences over $\mathbb{Q}$ with finite support.

The following set is closed in $\mathbb{R}^V$, so it is polish:

$$\mathcal{P} = \{\mu \in \mathbb{R}^V : \mu \text{ is a seminorm}\}$$

We can assign to **every element** $\mu \in \mathcal{P}$ a **separable Banach space** X_μ as follows:

1. Extend μ to $\bar{\mu}$, its unique seminorm extension over c_{00}.
2. Define the quotient norm $\tilde{\mu}$ over c_{00}/N_μ where
$$N_\mu = \{x \in c_{00} : \bar{\mu}(x) = 0\}.$$
3. $(X_\mu, \hat{\mu})$ will be any completion of $(c_{00}/N_\mu, \tilde{\mu})$.

In that way, the set $\bar{V} = \{v + N_\mu : v \in V\}$ is a dense set of X_μ (so it is separable).

Reciprocally, given a **separable Banach space** $(X, \|\cdot\|)$ with $\{x_n\}$ a dense sequence, we can **assign a seminorm** $\mu \in \mathcal{P}$ such that $X \equiv X_\mu$ by defining for every $v = \sum a_n e_n \in V$

$$\mu(v) = \left\| \sum_{n=1}^{\infty} a_n x_n \right\|.$$

This proves that $\mathcal{P}$ is a **codification** of the class of separable Banach spaces, where we can see V as a "universal dense" set for all separable Banach spaces.

Some known complexity classes

- The isometry class of ℓ_2 is closed.
- The isomorphy class of ℓ_2 is F_σ.
- The isometry class of ℓ_p with $1 \leq p < \infty$, $p \neq 2$, is $F_{\sigma\delta}$.
- The isometry class of $L_p[0,1]$ with $1 \leq p < \infty$, $p \neq 2$, is G_δ.

- Cúth M, Doležal M, Doucha M, Kurka O. *Polish spaces of Banach spaces.* Forum of Mathematics, Sigma. 2022;10:e26
- Cúth M, Doležal M, Doucha M, Kurka O. *Polish spaces of Banach spaces: Complexity of isometry and isomorphism classes.* J. Inst. Math. Jussieu. 2023:1-39.

Results on geometric classes

Diametral properties

A Banach space X has:

- The *local diameter 2 property* (LD2P) if every slice of B_X has diameter 2.
- The *diameter 2 property* (D2P) if every non-empty relatively weakly open subset of B_X has diameter 2.
- The *strong diameter 2 property* (SD2P) if every convex combination of slices of B_X has diameter 2.

Results (with Ginés López and Abraham Rueda)

- LD2P and SD2P classes are $F_{\sigma\delta}$.
- D2P class is G_δ-complete.
- The class of (locally, weakly) octahedral spaces is $F_{\sigma\delta}$.
- The class of uniformly convex spaces is $F_{\sigma\delta}$.