Projective characterizations of Lindenstrauss and Gurariĭ spaces

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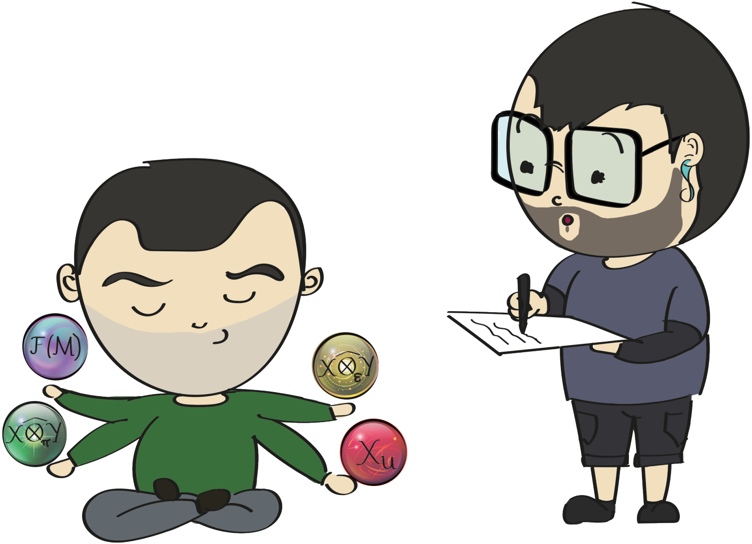
$\mathcal{F}(M)$
$X\hat{\otimes}_\varepsilon Y$
$X\hat{\otimes}_\pi Y$
X_u

Hahn-Banach on steroids

Hahn-Banach Theorem

Let Z be a Banach space, $Y \subset Z$ a closed subspace and $t : Y \to \mathbb{R}$ a bounded functional. There exists a linear bounded extension $T : Z \to \mathbb{R}$ of t such that $\|T\| = \|t\|$.

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Can we change $\mathbb{R}$ with some other Banach space?

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Injective spaces

X is **injective** if for every pair of Banach spaces $Y \subset Z$ and every bounded operator $t : Y \to X$, there exists a linear bounded extension $T : Z \to X$ such that $\|T\| = \|t\|$.

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Equivalent definitions of an injective space X

- X is injective.

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Equivalent definitions of an injective space X

- X is injective.
- For any $U \supset X$, X is 1-complemented in U.

Relaxing the definition

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X is ~~**injective**~~ if for every pair of finite dimensional Banach spaces $Y \subset Z$ and every bounded operator $t : Y \to X$, there exists a linear bounded extension $T : Z \to X$ such that $\|T\| = \|t\|$.

Relaxing the definition

Injective spaces

X is **injective** if for every pair of finite dimensional Banach spaces $Y \subset Z$, every $\varepsilon > 0$ and every bounded operator $t : Y \to X$, there exists a linear bounded extension $T : Z \to X$ such that $\|T\| \leq (1 + \varepsilon)\|t\|$.

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Lindenstrauss spaces

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Equivalent definitions of a Lindenstrauss space X

- X is a Lindenstrauss space.
- X is an L_1 predual.
- X^{**} is injective.
- For any $U \supset X$, X is an ideal in U.

Relaxing the definition

$Y \subset X$ is an **ideal** if for any $E \subset X$ of finite dimension and $\varepsilon > 0$, there exists $P : E \to Y$ such that:

- $P(e) = e$ for any $e \in E \cap Y$,
- $\|P\| \leq 1 + \varepsilon$.

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Gurariĭ spaces

X is a (strong) **Gurariĭ space** if for every pair of finite dimensional Banach spaces $Y \subset Z$, every $\varepsilon > 0$ and every isometry $t : Y \to X$, there exists a linear bounded extension $T : Z \to X$ such that it is an ε-isometry (isometry).

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Gurariĭ spaces

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Equivalent definitions of a Gurariĭ space X

- X is a Gurariĭ space.
- For any $U \supset X$, X is an almost isometric ideal in U.

Relaxing the definition

$Y \subset X$ is an **almost isometric ideal** if for any $E \subset X$ of finite dimension and $\varepsilon > 0$, there exists $P : E \to Y$ such that:

- $P(e) = e$ for any $e \in E \cap Y$,
- P is an ε-isometry.

Gurariĭ spaces

X is a (strong) **Gurariĭ space** if for every pair of finite dimensional Banach spaces $Y \subset Z$, every $\varepsilon > 0$ and every isometry $t : Y \to X$, there exists a linear bounded extension $T : Z \to X$ such that it is an ε-isometry (isometry).

Equivalent definitions of a Gurariĭ space X

- X is a Gurariĭ space.
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Not so relaxed

κ injective

X is a κ **injective** space if for every pair of Banach spaces $Y \subset Z$ with $\mathrm{dens}(Z) < \kappa$ and every bounded operator $t : Y \to X$, there exists a linear bounded extension $T : Z \to X$ such that $\|T\| = \|t\|$.

Not so relaxed

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Is there a projective characterization of these spaces?

Not so relaxed

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κ ideal

$Y \subset X$ is a κ **ideal** in X if for any $E \subset X$ with $\text{dens}(E) < \kappa$, there exists a norm 1 operator $P : E \to Y$ such that $P(e) = e$ for any $e \in E \cap Y$.

Not so relaxed

Equivalent definitions of κ injective spaces

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- For any $U \supset X$, X is a κ ideal in U.
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- X is an L_1 predual and it is a κ ideal in X^{**}.

Not so relaxed

Spaces of (almost) universal disposition

X is a space of **(almost) universal disposition for density** κ ((A)UD$_{<\kappa}$) if for every pair of Banach spaces $Y \subset Z$ with dens$(Z) < \kappa$ (every $\varepsilon > 0$) and every isometry $t : Y \to X$, there exists a linear bounded extension $T : Z \to X$ such that it is an (ε-)isometry.

Not so relaxed

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κ (almost) isometric ideal

$Y \subset X$ is a κ **(almost) isometric ideal** in X if for any $E \subset X$ with $\text{dens}(E) < \kappa$ (and $\varepsilon > 0$), there exists an (ε-)isometry $P : E \to Y$ such that $P(e) = e$ for any $e \in E \cap Y$.

Not so relaxed

Equivalent definitions of (A)UD$_{<\kappa}$ spaces

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Equivalent definitions of (A)UD$_{<\kappa}$ spaces

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- For any $U \supset X$ with U an (A)UD$_{<\kappa}$ space, X is a κ (almost) isometric ideal in U.

- $Y \subset X$ is an ideal iff there exists a Hahn–Banach extension operator $\varphi : Y^* \to X^*$.

Looking outside Lindenstrauss and Gurariĭ spaces

- $Y \subset X$ is an ideal iff there exists a Hahn–Banach extension operator $\varphi : Y^* \to X^*$.
- $Y \subset X$ is an ideal iff for every Z and $t : Y \to Z^*$ there exists a norm-preserving extension $T : X \to Z^*$.

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- $Y \subset X$ is an ideal iff Y^{**} is 1-complemented in X^{**}.
- If $Y \subset X$ is an ideal, then $Y \hat{\bigotimes}_\pi Z$ is a subspace of $X \hat{\bigotimes}_\pi Z$.

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- $Y \subset X$ is an ideal iff Y^{**} is 1-complemented in X^{**}.
- If $Y \subset X$ is an ideal, then $Y \hat{\bigotimes}_\pi Z$ is a subspace of $X \hat{\bigotimes}_\pi Z$.
- (Sims-Yost) Given $Y \subset X$ there exists $Y \subset Z \subset X$ such that Z is an ideal in X and $\text{dens}(Z) = \text{dens}(Y)$.

Looking outside Lindenstrauss and Gurariĭ spaces

- X has the LD2P (D2P, SD2P, DP) iff every ideal in X has the LD2P (D2P, SD2P, DP).

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- Every Banach space X is an almost isometric ideal in its bidual.
- (Abrahamsen) Given $Y \subset X$ there exists $Y \subset Z \subset X$ such that Z is an almost isometric ideal in X and $\text{dens}(Z) = \text{dens}(Y)$.

Looking outside Lindenstrauss and Gurariĭ spaces

- If Y is not Asplund and κ injective and X is an injective space that contains it, then $Y \hat{\bigotimes}_\pi Y$ is a κ ideal of $X \hat{\bigotimes}_\pi X$, but $Y \hat{\bigotimes}_\pi Y$ is not an L_1 predual (in particular it is not κ injective).

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- If $Y \subset X$ is a κ ideal, then $\mathcal{F}(Y)$ is a κ ideal of $\mathcal{F}(X)$, but $\mathcal{F}(Y)$ is not an L_1 predual.

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- If $Y \subset X$ is a κ ideal, then $\mathcal{F}(Y)$ is a κ ideal of $\mathcal{F}(X)$, but $\mathcal{F}(Y)$ is not an L_1 predual.
- If $J \subset I$ are sets and $\aleph_1 \leq \kappa \leq |J| < |I|$, then $\ell_p(J)$ is a κ isometric ideal in $\ell_p(I)$ for $1 < p < \infty$, but $\ell_p(J)$ is not an $\mathrm{AUD}_{<\kappa}$ space.

Auto-publicity

For more information and many more interesting results about κ (almost isometric) ideals look at:

E. Martínez Vañó and A. Rueda Zoca: *Transfinite (almost isometric) ideals in Banach spaces*, Arxiv preprint, 2025

Thank you!